Partially wrapped Fukaya categories & cobordism attachment

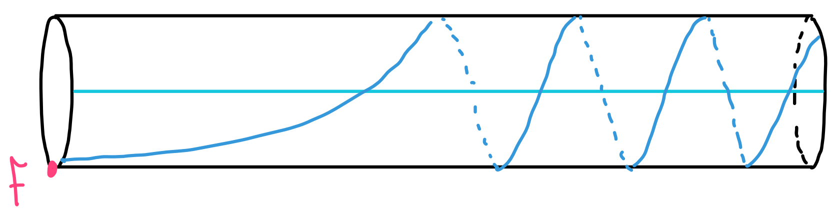
§1. Partially wrapped Floer cohomology

Conventions.

- X Liouville sector, $\partial_\infty X$ its contact bdy
- $\mathfrak{f} \subseteq \partial_\infty X$ a "stop"
- All Lagrangians L are <u>cylindrical</u> and <u>exact</u> (so $\partial_\infty L \subseteq \partial_\infty X$ is Legendrian)

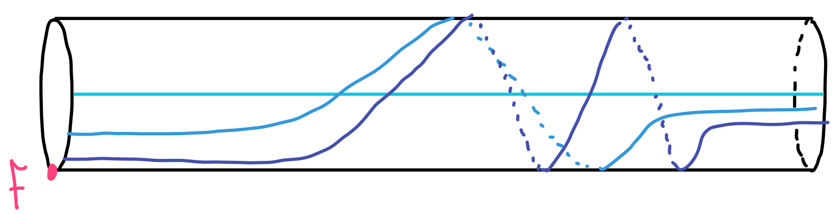
Preview. $L, K \subseteq X$ Lag. Let $L \rightsquigarrow L^w$ be a "large" & "positive" isotopy which avoids $\mathfrak{f}$ at ∞ (note L^w is not cylindrical). Then

$$HW^\bullet(L,K)_{X,\mathfrak{f}} := HF^\bullet(L^w, K)$$



Equivalently, consider a "cofinal" sequence of "positive" isotopies $L = L^{(0)} \rightsquigarrow L^{(1)} \rightsquigarrow L^{(2)} \rightsquigarrow \cdots$ (now $L^{(i)}$ are cylindrical) and define:

$$HW^\bullet(L,K)_{X,\mathfrak{f}} := \varinjlim_i HF^\bullet(L^{(i)}, K).$$



Rmk. HW is often defined by pushing L along a Ham. isotopy. This is equivalent to the isotopy consisting of exact Lagrangians (see Convention). This will be more convenient b/c we are wrapping partially.

Def. An isotopy $L_0 \rightsquigarrow L_1$ is *positive* if $\frac{d}{dt} \partial_\infty L_t$ lies in the positive side of the contact distribution.

Fact. If $L \rightsquigarrow L^w$ is a small positive isotopy, $HF^\bullet(L^w, L) \cong H^\bullet(L)$.

Def. Given a positive isotopy $L_0 \rightsquigarrow L_1$, define the *continuation element* $c(L_t) \in HF^\bullet(L_1, L_0)$:

- If $L_0 \rightsquigarrow L_1$ is small, let $c(L_t)$ be the image of $H^\bullet(L_0)$
- Otherwise, break $L_0 \rightsquigarrow L_1$ into small isotopies and compose the resulting continuation elts.

Given another Lag. K, composition w/ $c(L_t)$ defines a *continuation map*

$$HF^\bullet(L_0, K) \longrightarrow HF^\bullet(L_1, K).$$

Def. The *positive wrapping category* $(L \rightsquigarrow -)^+_{X,\mathfrak{f}}$ is the cat:

- Objects are isotopies $L \rightsquigarrow L^w$
- Morphisms b/t $L \rightsquigarrow L^w$, $L \rightsquigarrow L^{w'}$ are htpy classes of positive isotopies $L^w \rightsquigarrow L^{w'}$ s.t. $L \rightsquigarrow L^w \rightsquigarrow L^{w'}$ & $L \rightsquigarrow L^{w'}$ are htpic.

All isotopies above avoid $\mathfrak{f}$ at ∞.

Fact. $(L \rightsquigarrow -)^+_{X,\mathfrak{f}}$ is filtered and has countable cofinality. In particular, we can find $L = L^{(0)} \rightsquigarrow L^{(1)} \rightsquigarrow L^{(2)} \rightsquigarrow \cdots$ s.t. any $L \rightsquigarrow L^w$ admits a morphism to $L \rightsquigarrow L^{(i)}$ for i suff. large.

Def. $HW^\bullet(L,K)_{X,\mathfrak{f}} := \varinjlim_{(L \rightsquigarrow L^w)^+} HF^\bullet(L^w, K) = \varinjlim_i HF^\bullet(L^{(i)}, K)$.

cofinal seq.

§2. The partially wrapped Fukaya category

Def. Let I be a countable set of Lag. containing one Lag. from each isotopy class. For each $L \in I$, choose a cofinal sequence

$$L = L^{(0)} \rightsquigarrow L^{(1)} \rightsquigarrow L^{(2)} \rightsquigarrow \cdots$$

Let $\mathcal{O} = \{L^{(i)} \mid L \in I, i \in \mathbb{Z}_{\geq 0}\}$. This is a poset w/ partial order $L^{(i)} \leq K^{(j)}$ iff $i \leq j$. Choose everything s.t. every totally ordered collection of Lag are mutually $\pitchfork$.

Define $\mathcal{O}$ as an A_∞-category

$$\mathrm{Hom}_{\mathcal{O}}(L, K) = \begin{cases} CF^\bullet(L, K) & L > K \\ \mathbb{Z} & L = K \\ 0 & \end{cases}$$

To define the A_∞-operations, count hol. disks w.r.t. a choice of Floer data. Note $\pitchfork$'lity of Lag. always holds!

Def. Let C be the set of continuation elts $c \in HF^0(L^{(i+1)}, L^{(i)})$.
Let $\mathcal{W}(X, \mathfrak{f}) := \mathcal{O}[C^{-1}]$ (localization of A_∞-categories).

§3. Cobordism Attachment

Thm. (Prop 1.37 in GPS2) $L, K \subseteq X$ disj. Lag (disj. from $\mathfrak{f}$) whose primitives vanish at ∞. Let $C \subseteq \mathbb{R} \times \partial_\infty X$ an exact Lag. cobordism (disj. from $\mathfrak{f}$) w/ negative end $\partial_\infty L \sqcup \partial_\infty K$ s.t. its primitive f_C satisfies

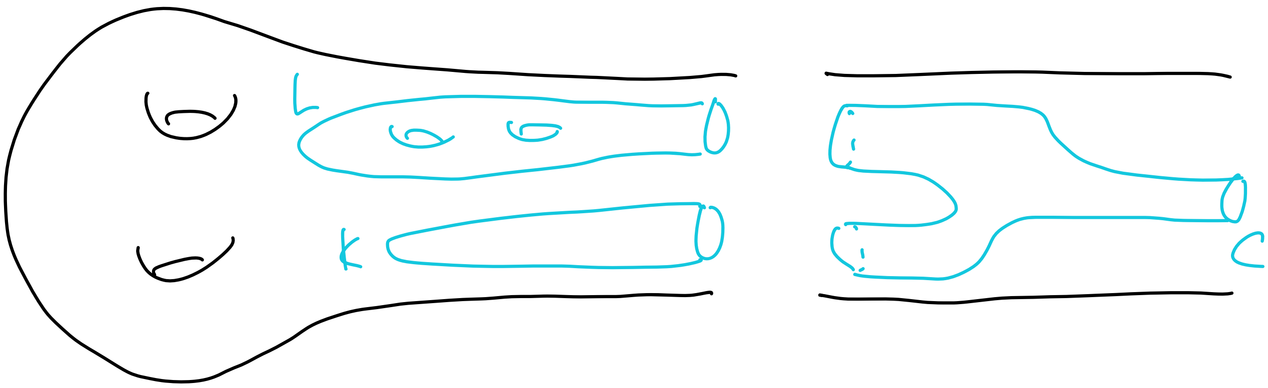
$$f_C|_{\partial_\infty L} < f_C|_{\partial_\infty K}.$$

If the image of C in $\partial_\infty X$ is "thin", meaning any $K \subseteq X$ disj. from

F can be wrapped to avoid C, then:

$$L \#_C K \simeq \mathrm{Cone}(L \to K).$$

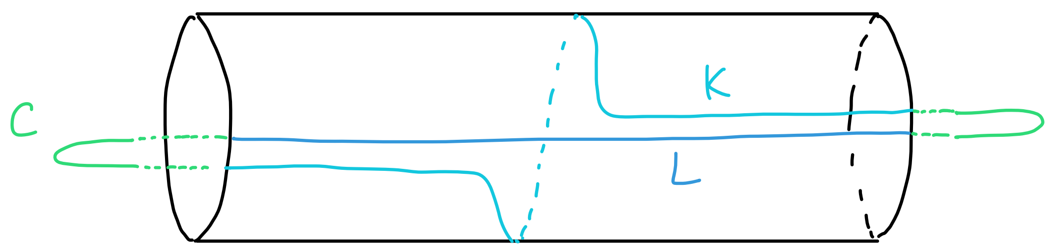
attach C to $L \sqcup K$



Def. For an A_∞-category $\mathcal{C}$, $Z \simeq \mathrm{Cone}(X \to Y)$ if

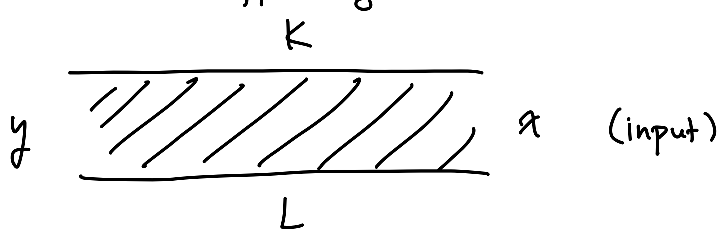
$$\mathcal{C}(A,Z) \simeq \mathrm{Cone}(\mathcal{C}(A,X) \longrightarrow \mathcal{C}(A,Y)).$$

Ex.



$$\Rightarrow \quad S^1 = L \#_C K \simeq [L \to K] = [L \to L].$$

Review. Consider a hol. disk appearing in the diff. of $CF^\bullet(L,K)$:



Stokes $\Rightarrow$ $E(u) = \int u^*\omega = \int (\partial u)^*\lambda = (f_K(y) - f_K(x)) + (f_L(x) - f_L(y))$

$$= (f_L(x) - f_K(y)) - (f_L(y) - f_K(y)) = \mathcal{A}(x) - \mathcal{A}(y).$$

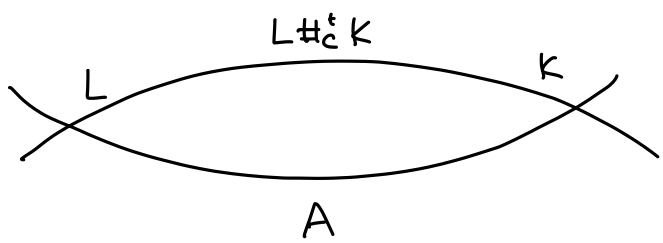
proof of Thm. Let $L \#_C^t K$ be the result of attaching the t-translate of C. Test $L \#_C^t K$ against a Lag A. For t large, $A \cap (L \#_C^t K) = (A \cap L) \sqcup (A \cap K)$. So

$$CF^\bullet(A, L \#_C^t K) = CF^\bullet(A, L) \oplus CF^\bullet(A, K)$$

as abelian groups. So $d = \begin{pmatrix} d_{LL} & d_{KL} \\ d_{LK} & d_{KK} \end{pmatrix}$. We WTS $d_{LL} = d_L$, $d_{KK} = d_K$, $d_{KL} = 0$. Observe:

- As we change t (i.e., translate C), the constants $f_C|_{\partial_\infty L}$, $f_C|_{\partial_\infty K}$ grow exponentially. In particular, the difference $f_C|_{\partial_\infty K} - f_C|_{\partial_\infty L} \to \infty$.

Now consider disks counted by d_{KL}:



For t large $\int (\partial u)^* \lambda$ = stuff indep. of t + $f|_{\partial_\infty L} - f_C|_{\partial_\infty K} \to -\infty$, so the energy is negative! So $d_{KL} = 0$. For d_{LL}, the energy of curves does not change w/ t. By monotonicity, these curves are a priori bdd away from ∞ $\Rightarrow$ the only curves that show up are those contributing to d_L. Hence, $d_{LL} = d_L$, $d_{KK} = d_K$.

There are two issues remaining:

① We are considering CF & not CW.

② We cannot choose t large enough s.t. the above argument works $\forall A$.

These can be resolved through some algebraic tricks which we will not discuss. ∎