McLean's theorem on the quantum cohomology of birational Calabi-Yau manifolds

The goal of these notes is to explain some of the ingredients that go into the proof of following result of McLean.

Theorem 1 ([10])**.** *Birational Calabi-Yau manifolds have isomorphic quantum cohomology algebras up to a change of Novikov rings.*

By a *Calabi-Yau manifold*, we mean a smooth complex projective variety X with $c_1(K_X) = 0$, where K_X denotes the canonical line bundle of X. In particular, the result does not require that X be simply connected, or K_X be trivial as a holomorphic line bundle.

Two varieties $X, \check{X}$ are *birational* if there exist Zariski open subsets $U \subset X$ and $\check{U} \subset \check{V}$ such that U and $\check{U}$ are isomorphic.

By quantum cohomology, we always mean small quantum cohomology. Fix any field $\mathbb{K}$ and any Kähler class $[\omega] \in H^2(X;\mathbb{R})$, thus determining a Novikov ring Λ_ω. *Quantum cohomology* $QH(X;\Lambda_\omega)$ is an algebra over Λ_ω, which as a module is isomorphic to $H^*_{\text{sing}}(X;\Lambda_\omega)$. Since the complex structure of X is fixed, quantum cohomology of X is well-defined up to a change of Novikov rings. It is in this sense that Theorem 1 should be understood.

More precisely, if we fix Kähler classes $[\omega]$ on X and $[\check{\omega}]$ on $\check{X}$, then Theorem 1 says that there exists some algebra over the ring $\Lambda_{\omega,\check{\omega}} := \Lambda_\omega \cap \Lambda_{\check{\omega}}$, which we shall suggestively denote by $\mathbb{SH}$, such that we have the following isomorphisms of algebras

$$QH(X;\Lambda_\omega) \cong \mathbb{SH} \otimes_{\Lambda_{\omega,\check{\omega}}} \Lambda_\omega,$$
$$QH(\check{X};\Lambda_{\check{\omega}}) \cong \mathbb{SH} \otimes_{\Lambda_{\omega,\check{\omega}}} \Lambda_{\check{\omega}}.$$

To better appreciate this result, let us examine the following immediate corollary.

Corollary 2. *Birational Calabi-Yau manifolds have equal Betti numbers.*

This is quite remarkable, since examples show that birational Calabi-Yau manifolds need not be homotopy equivalent [2, Example 7.7]. So there is no "apparent reason" why two such varieties must have the same Betti numbers.

In fact, in this example, it is precisely from comparing the cup products on singular cohomology that we infer the two varieties are not homotopy equivalent. This means that somehow the quantum corrections in quantum cohomology perfectly cancel out the change in the classical cup products. This was observed by Morrison [11] and Ruan [13].

Corollary 2 was previously obtained by Batyrev [1], Kontsevich, et al. using methods drastically different from McLean's. In fact, their methods produce sharper results. A very accessible introduction to this theory is [12].

As for quantum cohomology, earlier works include Li-Ruan [8], which proves Theorem 1 for Calabi-Yau 3-folds, and Lee-Lin-Qu-Wang [5, 6, 7], which solves the case of "ordinary flops", which is a special kind of birational map. The latter result does not need to assume that the two manifolds have $c_1 = 0$; it suffices that they have "equivalent" canonical bundles in some sense. Moreover, they obtain an isomorphism of big quantum cohomology algebras, identified through a natural correspondence map. However, very little is known about the structure of general birational maps between Calabi-Yau manifolds.

We will now proceed to explain McLean's proof of Theorem 1. In the author's opinion, this is one of the deepest proofs in the field of symplectic geometry. As we shall see, the proof fails at almost every step, and yet at every step a new idea is found to solve the problem. The problem would then migrate to a different place, where a different idea is needed. At the end, everything miraculously converges, and the theorem is proved.

The basic idea behind the proof is an old proposal due to Seidel [14], viewing quantum cohomology of a closed symplectic manifold as a deformation of symplectic cohomology of the complement of a divisor. Its relevance to the problem at hand is through the following algebro-geometric fact: any two birational Calabi-Yau manifolds are isomorphic as algebraic varieties outside subvarieties of complex codimension at least two. This is true because otherwise the birational map would have to contract some divisor, which must be reflected in the first Chern class. We pick divisors containing these codimension two subvarieties, so the complements of the divisors are isomorphic as algebraic varieties. By Seidel's principle, the two quantum cohomology algebras should be deformations of the same object. Moreover, the deformations should be given by certain counts of holomorphic discs or planes, which after generic perturbation should avoid subsets of real codimension four. Hence the two Calabi-Yau manifolds should have isomorphic quantum cohomology algebras.

Unfortunately, there is no result in the literature making precise Seidel's idea in the generality that is needed (this turns out to be an extremely subtle problem). On the other hand, there is an approximation of the idea which applies in great generality. This is done using the S-shaped Hamiltonians that were introduced to define the Hofer-Zehnder capacity [4].

Definition 3. Let K be a subset in a closed symplectic manifold (X, ω). Let $\{H_n\}$ be a sequence of non-degenerate Hamiltonians on X with the following properties.

1. $H_n \leq H_{n+1}$.
2. $\lim_{n\to\infty} H_n = \begin{cases} 0 & \text{on } K, \\ +\infty & \text{on } X \setminus K. \end{cases}$

We then define

$$SH(K \subset X) := \varinjlim_{a\to-\infty} \varprojlim_{b\to+\infty} \varinjlim_{n\to\infty} HF_a^b(H_n).$$

Here $HF_a^b(H_n)$ denotes the truncated Floer cohomology group generated by those orbits with action in the interval $[a, b)$. For the action functional to make sense, we enlarge the Floer chain complex so that the generators are orbits equipped with a relative homology class. By standard continuation map arguments, $SH(K \subset X)$ is independent of the choice of sequence $\{H_n\}$.

We will only be interested in the case where K is a Liouville domain embedded symplectically in X. In [10] the invariant is called symplectic cohomology, but to avoid confusion with Viterbo's symplectic cohomology associated to a Liouville manifold, we will call this *quantitative* symplectic cohomology.

To see how this is relevant to Seidel's idea, let us consider which generators in the Floer chain complex survive under the limiting procedure. Given an S-shaped Hamiltonian, one can set things up so that there are four kinds of orbits:

- Morse critical points in K.
- Two copies of each Reeb orbit in the region where H is convex.
- Two copies of each Reeb orbit in the region where H is concave.
- Morse critical points in $X \setminus K$.

See Figure 1.

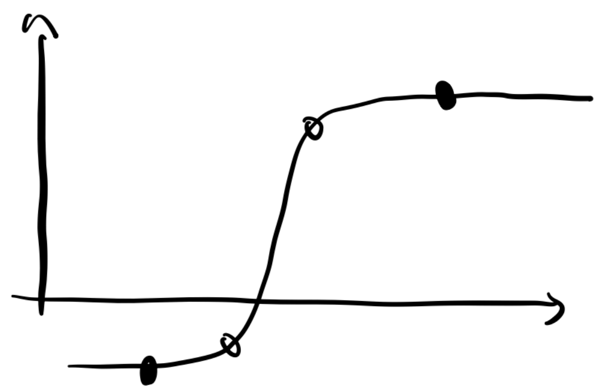

Figure 1: Four kinds of orbits of an S-shaped Hamiltonian.

The first two sets of orbits are the usual generators for Viterbo's symplectic cohomology (ignoring for the moment the fact that our generators are equipped with relative homology classes). As the Hamiltonian tends to $+\infty$ outside K, the Morse critical points outside K have action $\to +\infty$, hence never appear in truncated Floer cohomology. This is the crucial property that we want.

To make this argument rigorous, McLean introduces variants of the standard action functional, and uses Stokes theorem in an appropriate way to show that the differentials and continuation maps respect these various action functionals. By playing around with the different filtrations, as well as using a certain property that relies on the $c_1 = 0$ condition (we will come back to this later), he is able to show that the critical points in $X \setminus K$ never contribute to $SH(K \subset X)$, no matter what homology class it is paired with.

As for the third set of orbits, one just has to bear with them. They are annoying, but it turns out that in the end they do not ruin the proof.

However, we now have a different problem. Is quantitative symplectic cohomology still isomorphic to quantum cohomology? It turns out that this is not true in general. For example, take X to be the 2-sphere, and K a disc. If K has area strictly less than half of the total area, then $SH(K \subset X) = 0$. Otherwise, $SH(K \subset X)$ is isomorphic to $QH(X)$. (Exercise.) In response to this, McLean identifies a sufficient condition for SH to be isomorphic to QH.

Proposition 4. *If $\overline{X \setminus K}$ is stably Hamiltonian displaceable, then*

$$SH(K \subset X) \cong QH(X).$$

A subset $D \subset M$ is stably Hamiltonian displaceable if $D \times S^1$ is Hamiltonian displaceable in $M \times T^*S^1$. There are no topological obstructions to being stably displaceable.

This result was independently obtained by Varolgunes in [15].

The proof of Proposition 4 uses ideas of Ginzburg [3]. It is quite technical, so we will only give a rough sketch. In fact, we will only treat the easier "dual" statement: if K itself is Hamiltonian displaceable, then $SH(K \subset X) = 0$. Let H_{disp} be the Hamiltonian which supposedly displaces K from itself. We modify the sequence of Hamiltonians in the definition of $SH(K \subset X)$ by concatenating with H_{disp}. This means that one first flows half time along X_{H_n}, and then half time along $X_{H_{disp}}$. By cofinality this does not change $SH(K \subset X)$. However, the only orbits of the concatenated Hamiltonians lie away from K, since $X_{H_{disp}}$ displaces K from itself. And these orbits have the wrong action. So $SH(K \subset X) = 0$. This argument is not rigorous, but the illustrious reader may be able to make a correct proof out of it.

Even with this in hand, our goal still seems very distant. How can we tell when a subset is stably Hamiltonian displaceable? Certainly having zero volume is not sufficient; consider the equator of the 2-sphere, which is an essential Lagrangian. We also challenge the reader to classify the subsets in the 2-torus that are (stably) Hamiltonian displaceable. This seems to be a difficult problem.

McLean proves the following very sharp result.

Proposition 5. *A stratified subset consisting of symplectic submanifolds of codimension at least two is stably Hamiltonian displaceable. In particular, a divisor in a Kähler manifold is stably Hamiltonian displaceable.*

Hence if we let K be a sufficiently large subset in the complement of the divisor, then $SH(K \subset X) \cong QH(X)$.

The proof of Proposition 5 is quite elegant. In $M \times T^*S^1$, the constant vector field V pointing along a cotangent fiber direction clearly displaces $D \times S^1$ from itself. However, it is certainly not Hamiltonian. To fix this, we look for a "curled up" symplectic embedding ι of $D \times T^*S^1$ into $M \times T^*S^1$. This means that the image of ι is compact. Then the vector field $V - (\varphi_V^t)_* \iota_* V$ is Hamiltonian and still eventually displaces $D \times S^1$, since $\iota_* V$ has compact support. See Figure 2. To construct this ι, one proceeds by stratum-wise induction, and at each step one extends the symplectic embedding by applying an h-principle and a Moser argument.

There is still one more problem that we have not addressed. The two divisor complements are isomorphic as algebraic varieties, but their symplectic structures could be very different. In fact, in general one cannot expect the ample cones (of the projective varieties or of the affine varieties) to be much related. For example, if two birational Calabi-Yau manifolds have a common resolution on which the pull-backs of two ample line bundles coincide, then they have to be isomorphic in the first place. (Exercise.) This seems quite problematic, as Hamiltonian Floer cohomology does depend on the symplectic structure.

McLean solves this by gluing symplectic forms.

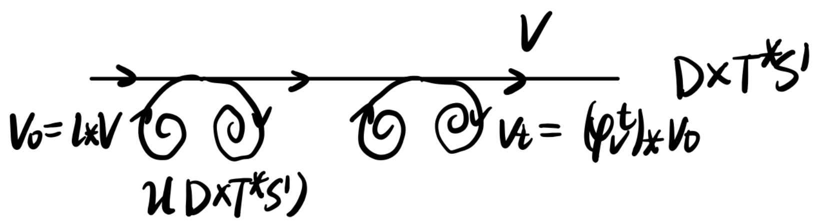

Figure 2: Construction of the displacing Hamiltonian vector field.

Proposition 6. *Suppose ω_0 and ω_1 are Kähler forms on the two birational Calabi-Yau manifolds X_0 and X_1. Then there exists some constant $C > 0$ and a Kähler form on X_0 which is equal to $C\omega_0$ near the divisor in X_0 and (the pull-back of) ω_1 near the skeleton of the divisor complement of X_0.*

In general, one cannot just glue symplectic forms. However, in this case, one can work on the level of the plurisubharmonic functions given by the ample line bundles. Say we have two functions ρ_0, ρ_1. Subtracting by a large constant, we may assume that ρ_0 is positive near the divisor and negative near the skeleton. Then we can find some constant $C > 0$ such that $C\rho_0 > \rho_1$ near the divisor and $C\rho_0 < \rho_1$ near the skeleton. Now we take $\max\{C\rho_0, \rho_1\}$. Even though this function is merely continuous, it still satsfies the weak mean value property, which characterizes plurisubharmonicity. By mollification, we obtain a smooth plurisubharmonic function ρ which interpolates between $C\rho_0$ and ρ_1. The desired Kähler form is $-dd^c\rho$ in the divisor complement and $C\omega_0$ in the divisor. See Figure 3. Using the continuation map equation with this glued symplectic form, we can define a continuation map from $SH(K_0 \subset X_0)$ to $SH(K_1 \subset X_1)$.

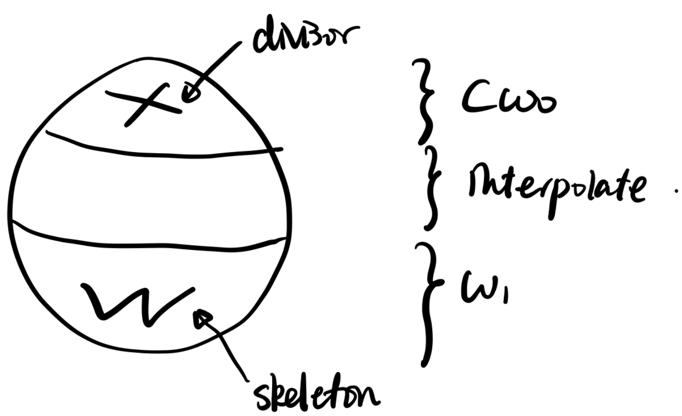

Figure 3: The glued Kähler form.

By doing this, we have introduced a new problem (this seems to be the theme of the proof). Multiplying ω_0 by C is effectively making the cobordism $K_0 \setminus K_1$ very long. This works against our wish that the complements of K_0 and K_1 are stably Hamiltonian displaceable, so that we may use Proposition 4 to recover quantum cohomology.

To deal with this, McLean shows that by choosing ∂K_0 and ∂K_1 nicely, one can ensure that $SH(K_0 \subset X_0) \to SH(K_1 \subset X_1)$ is an isomorphism. More precisely, we want both contact boundaries to be *index bounded*, i.e., there are only finitely Reeb orbits in each degree. This means that infinite sums of orbits can be ignored; then it is not too difficult to show that continuation maps induce an isomorphism.

Proposition 7. *If ∂K_0 and ∂K_1 are index bounded, then continuation maps induce an isomorphism $SH(K_0 \subset X_0) \to SH(K_1 \subset X_1)$.*

In fact, this condition is also used to rule out the constant orbits in $X \setminus K$, as we discussed earlier. However, index boundedness is a rather strong restriction. For example, the contact boundary of $\mathbb{C}^* \times \mathbb{C}^*$ is never index bounded. (Exercise.) Even the example $\mathbb{C} \subset S^2$ is not index bounded, once we account for the corrections to the index coming from the relative homology classes.

In our case, because $c_1(X) = 0$, the index of an orbit is well-defined, i.e., it does not depend on the choice of relative homology class. McLean shows that we can always find index bounded contact hypersurfaces in this case. To make the relevant index calculation, we pass to some resolution so that the divisor is normal crossings. Then if we fix a trivialization of the tangent bundle of K, a smooth section which is non-vanishing on K vanishes to non-negative orders along the divisor (this uses $c_1(X) = 0$). We choose a nice contact form so that the Reeb orbits wind around a standard neighborhood of the normal crossings divisor, which is possible by earlier work of McLean [9]. Then one can make a local calculation to find that the indices of the orbits are proportional to their winding numbers along the divisor. Since the length of an orbit is also proportional to its winding numbers, this means that the contact form is index bounded.

Proposition 8. *If $c_1(X) = 0$, then given any ample divisor in X, one can construct index bounded contact hypersurfaces surrounding it.*

Now we have all the ingredients for the proof. We leave it to the reader to assemble the ingredients together.

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