**A DYNAMICIST'S PROOF OF GROMOV'S NONSQUEEZING**

**Theorem 1** ([2])**.** *Suppose there is a symplectic embedding*

$$B(R) = \{(x,y) \in \mathbb{R}^{2n} \mid |x|^2 + |y|^2 < R^2\} \to Z(r) = \{(x,y) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 < r^2\}.$$

*Then* $R \leq r$.

Note that if we instead took $\hat{Z}(r) = \{(x,y) \in \mathbb{R}^{2n} \mid x_1^2 + x_2^2 < r^2\}$, then there is always a linear symplectic embedding $B(R) \to \hat{Z}(r)$.

Gromov's celebrated nonsqueezing theorem admits the following dynamical explanation [4]. (As an aside, Hofer and Zehnder's book [4] is very beautifully written and I strongly recommend.)

Observe that all characteristics on $\partial B(R)$ are closed, have period $2\pi$, and action $\pi R^2$. All characteristics on $\partial Z(r)$ are closed, wind around the cylinder, and have action $\pi r^2$. In contrast, characteristics on $\partial \hat{Z}(r)$ are straight lines. Imagine there is an "optimal" embedding $B(R) \to Z(r)$, in which case one is inclined to think that the boundary of the embedded ball would touch the boundary of the cylinder. Let's say this is happening along a closed characteristic. Then by the invariance of action under symplectic maps, we should have $r = R$.

However, a rigorous proof of Gromov's nonsqueezing theorem along these lines does not exist (according to Hofer and Zehnder).

**Definition 2.** A *symplectic capacity* is a map

$$(M,\omega) \mapsto c(M,\omega) \in [0,\infty]$$

where $(M,\omega)$ is a symplectic manifold (possibly with boundary) of fixed dimension $2n$, satisfying the axioms (A1)–(A3).

- A1. Monotonicity: $c(M,\omega) \leq c(N,\tau)$ if there exists a symplectic embedding $(M,\omega) \to (N,\tau)$.
- A2. Conformality: $c(M,\alpha\omega) = |\alpha| c(M,\omega)$ for $\alpha \neq 0$.
- A3. Nontriviality: $c(B(1),\omega_0) = \pi = c(Z(1),\omega_0)$.

Let $c$ be a symplectic capacity. By (A3) and (A2),

$$c(B(r)) = r^2 c(B(1)) = \pi r^2,\ c(Z(R)) = R^2 c(Z(1)) = \pi R^2.$$

Then Gromov's nonsqueezing follows from (A1).

Therefore, to prove Gromov's nonsqueezing, it suffices to show that there exists a symplectic capacity.

The goal of this note is to describe the construction of a symplectic capacity called the *Hofer-Zehnder capacity* [3]; in particular, to prove that it satisfies axioms (A1)–(A3).

The motivating idea is as follows. Consider $H$ of the following form (see Figure 1). If the oscillation of $H$ is sufficiently large, then $X_H$ has, independent of the size of its support, a fast periodic orbit, i.e., an orbit having small period, say $0 < T \leq 1$. The threshold that $\mathrm{Osc}(H)$ needs to pass in order to admit such a fast periodic orbit encodes symplectic information of $(M,\omega)$.

To be precise, let $\mathcal{H}(M,\omega)$ be the set of smooth functions $H$ satisfying the following properties.

1. For some compact set $K \subset M \setminus \partial M$ and constant $m(H)$, $H|_{M\setminus K} \equiv m(H)$.
2. For some open set $U \subset M$, $H|_U \equiv 0$.
3. $0 \leq H(x) \leq m(H),\ x \in M$.

A function $H \in \mathcal{H}(M,\omega)$ is *admissible* if all periodic orbits of $X_H$ are either constant or have period $> 1$. Denote the set of admissible functions by $\mathcal{H}_a(M,\omega)$. Define

$$c_0(M,\omega) = \sup\{m(H) \mid H \in \mathcal{H}_a(M,\omega)\}.$$

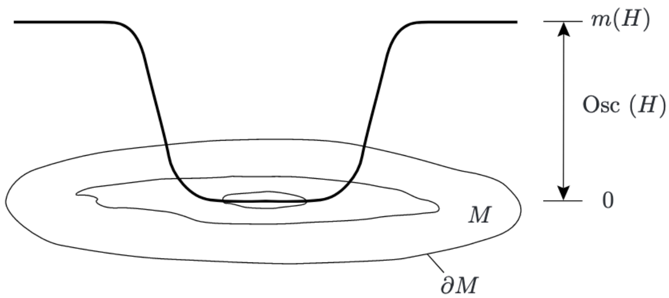


FIGURE 1. Reproduced from [4] for educational purposes only.

In other words, $c_0(M, \omega)$ is the infimum of numbers $a$ such that as soon as $m(H) > a$, the vector field $X_H$ possesses a nonconstant orbit of period $T$ for some $T \in (0, 1]$.

Now we must verify that $c_0$ satisfies the axioms (A1)–(A3). The first two are easy. (Exercise: prove it.)

**Lemma 3.** *$c_0(B(1), \omega_0) \geq \pi$.*

*Proof.* For every $0 < \varepsilon < \pi$, we will show that $c_0(B(1), \omega_0) \geq \pi - \varepsilon$ by constructing a function $H \in \mathcal{H}_a(B(1))$ with $m(H) = \pi - \varepsilon$. Choose a smooth function $f : [0, 1] \to [0, \infty)$ such that

(1) $0 \leq f' < \pi$,
(2) $f(t) = 0$ for $t$ near 0,
(3) $f(t) = \pi - \varepsilon$ for $t$ near 1.

Define $H(x) = f(|x|^2)$ for $x \in B(1)$. The Hamiltonian system

$$-J\dot{x} = \nabla H(x) = 2f'(|x|^2)x$$

has the function $G(x) = \frac{1}{2}|x|^2$ as an integral (i.e. conserved quantity), since $\langle \nabla G, J\nabla H \rangle = 0$. Therefore if $x(t)$ is a solution, then

$$2f'(|x(t)|^2) = a$$

is a constant and the solution satisfies $-J\dot{x} = ax$. Consequently, all solutions are periodic and are given by

$$x(t) = e^{aJt}x(0) = (\cos at)x(0) + (\sin at)Jx(0).$$

If $a = 0$ then the solution is constant, whereas if $a > 0$ then the solution has period $T = \frac{2\pi}{a} > 1$. □

Then it remains to prove that $c_0(Z(1), \omega_0) \leq \pi$. This is the difficult part.

**Theorem 4.** *If $H \in \mathcal{H}(Z(1))$ and $m(H) > \pi$, then $X_H$ has a nonconstant orbit of period $T = 1$.*

To prove this, we may assume without loss of generality that $H$ vanishes near the origin. (Exercise: prove this.) We extend $H$ to $\mathbb{R}^{2n}$ in the following way. Since $H \in \mathcal{H}(Z(1), \omega_0)$, there is an ellipsoid $E = E_N$ such that $H \in \mathcal{H}(E, \omega_0)$, where $E = \{z \in \mathbb{R}^{2n} \mid q(z) < 1\}$ and $q = x_1^2 + y_1^2 + \frac{1}{N^2}\sum_{j=2}^{n}(x_j^2 + y_j^2)$ for $N$ sufficiently large. Now for any $\varepsilon > 0$ such that $m(H) > \pi + \varepsilon$, pick a smooth function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following properties.

(1) $f(s) = m(H),\ s \leq 1$.
(2) $f(s) \geq (\pi + \varepsilon)s,\ s \in \mathbb{R}$, with equality for large $s$.
(3) $0 < f'(s) \leq \pi + \varepsilon,\ s > 1$.

We now define $\overline{H} : \mathbb{R}^{2n} \to \mathbb{R}$ by

$$\overline{H}(z) = \begin{cases} H(z) & z \in E \\ f(q(z)) & z \notin E. \end{cases}$$

Note that $\overline{H}$ is quadratic at infinity.

**Proposition 5.** *Assume $x(t)$ is a 1-periodic orbit of $X_{\overline{H}}$. If*

$$\Phi(x) = \int_0^1 \frac{1}{2}\langle -J\dot{x}, x\rangle - \overline{H}(x(t))dt > 0,$$

*then $x(t)$ is nonconstant and $x(t) \subset E$. Hence $x(t)$ is a nonconstant 1-periodic orbit of $X_H$ on $Z(1)$.*

*Proof.* If $x$ is a constant orbit, then $\Phi(x) \leq 0$. If $x(t_0) \notin E$ for some $t_0$, then since $X_{\overline{H}}$ vanishes on $\partial E$, $x(t) \notin E$ for all $t$, and hence it solves the equation

$$-J\dot{x} = \nabla \overline{H}(x) = f'(q(x))\nabla q(x).$$

The function $q$ is an integral of this equation since $\langle \nabla q, Jf'(q)\nabla q\rangle = 0$. Hence if $x(t)$ is a solution then

$$q(x(t)) = \tau$$

is constant in $t$, so we can compute

$$\begin{aligned}\Phi(x) &= \int_0^1 \frac{1}{2}\langle -J\dot{x}, x\rangle - \overline{H}(x(t))dt \\ &= \int_0^1 \frac{1}{2}f'(\tau)\langle \nabla q(x), x\rangle - f(q(x))dt \\ &= \int_0^1 f'(\tau)q(x) - f(q(x))dt \\ &= \int_0^1 f'(\tau)\tau - f(\tau)dt \\ &\leq 0.\end{aligned}$$

The third equality uses the fact that $q$ is a quadratic form. Thus the proposition is proved. □

We now change notation and replace $\overline{H}$ by $H$. Our goal becomes finding a 1-periodic orbit of $X_H$ which satisfies $\Phi(x) > 0$. To find this solution, we would like to use the following variational principle.

Consider the loop space $\Omega = C^\infty(S^1, \mathbb{R}^{2n})$. Consider the function $\Phi : \Omega \to \mathbb{R}$ given by

$$\Phi(x) = \int_0^1 \frac{1}{2}\langle -J\dot{x}, x\rangle - H(x(t))dt.$$

The critical points of this function are precisely the 1-periodic orbits of $X_H$. (Of course you know this.)

From a variational point of view, this action principle is highly degenerate. Indeed, consider the loops $x_m(t) = e^{m2\pi Jt}\xi$ where $|\xi| = 1$. One computes that $\int_0^1 \frac{1}{2}\langle -J\dot{x}_m, x_m\rangle dt = \pi m$, whereas the second part of $\Phi$ stays bounded. Therefore, the functional is bounded neither below nor above. In particular, the variational techniques based on minimizing sequences (which can be used e.g. to deduce the existence of closed geodesics on Riemannian manifolds) do not apply.

As we all know, these difficulties were surmounted in Floer's celebrated work [1], using methods similar to Gromov's. In this note, we will explain a different strategy due to Hofer-Zehnder, which in turn is based on earlier ideas of Rabinowitz [5].

The tool which we shall use to find the 1-periodic orbit is the following classical minimax lemma. Let $E$ be a Hilbert space, $f \in C^1(E, \mathbb{R})$, and $\mathcal{F}$ a family of subsets of $E$. We define the *minimax*

$$c(f, \mathcal{F}) = \inf_{F\in\mathcal{F}} \sup_{x\in F} f(x) \in \mathbb{R} \cup \{\pm\infty\}.$$

**Lemma 6** (Minimax lemma)**.** *Suppose that $f$ and $\mathcal{F}$ satsify the following conditions.*

(1) *$f$ satisfies the Palais-Smale condition, i.e., every sequence $x_j \in E$ satisfying*

$$\nabla f(x_j) \to 0 \text{ and } |f(x_j)| \leq C < \infty \text{ for some } C$$

*possesses a convergent subsequence.*

(2) $\dot{x} = -\nabla f(x)$ *defines a global flow* $\varphi^t(x)$*, i.e., the Cauchy initial value problem can be solved uniquely and for all times* $t \in \mathbb{R}$.
(3) *The family* $\mathcal{F}$ *is positively invariant under the flow, i.e., if* $F \in \mathcal{F}$ *then* $\varphi^t(F) \in \mathcal{F}$ *for all* $t \geq 0$.
(4) $c(f, \mathcal{F}) \in (-\infty, \infty)$.

*Then the real number* $c(f, \mathcal{F})$ *is a critical value of* $f$*, i.e., there exists* $x^* \in E$ *such that*

$$\nabla f(x^*) = 0 \text{ and } f(x^*) = c(f, \mathcal{F}).$$

*Proof.* It suffices to show that for every $\varepsilon > 0$ there exists an $x \in E$ such that

$$f(x) \in [c - \varepsilon, c + \varepsilon] \text{ and } \|\nabla f(x)\| \leq \varepsilon.$$

Indeed, choosing $\varepsilon_j = 1/j$, we find a sequence $x_j$ which has, by the P.S. condition, a convergent subsequence whose limit is the desired critical point. Now we argue by contradiction. Assume there exists $\varepsilon > 0$ such that $\|\nabla f(x)\| > \varepsilon$ for all $x$ satisfying $f(x) \in [c-\varepsilon, c+\varepsilon]$. By the definition of $c$, there exists $F \in \mathcal{F}$ such that $\sup_{x \in F} f(x) \leq c+\varepsilon$. Pick any $x \in F$, so $f(x) \leq c+\varepsilon$. We claim that $f(\varphi^{t^*}(x)) \leq c - \varepsilon$ for $t^* = 2/\varepsilon$. Indeed, assume by contradiction that $f(\varphi^t(x)) > c - \varepsilon$ for all $t \in [0, t^*]$. Then by assumption $\|\nabla f(\varphi^t(x))\| \geq \varepsilon$ for all $t \in [0, t^*]$, and hence

$$f(\varphi^{t^*}(x)) - f(x) = -\int_0^{t^*} \|\nabla f(\varphi^s(x))\|^2 ds \leq -\varepsilon^2 t^* = -2\varepsilon.$$

Now set $F^* = \varphi^{t^*}(F)$. We have shown that $\sup_{x \in F^*} f(x) \leq c - \varepsilon$. But $F^* \in \mathcal{F}$, a contradiction. □

To find the convenient Hilbert space, we represent loops $x \in C^\infty(S^1, \mathbb{R}^{2n})$ by their Fourier series

$$x(t) = \sum_{k \in \mathbb{Z}} e^{k 2\pi J t} x_k.$$

We compute

$$\begin{aligned}\int_0^1 \langle -J\dot{x}, y \rangle dt &= 2\pi \sum_{j \in \mathbb{Z}} j \langle x_j, y_j \rangle \\ &= 2\pi \sum_{j > 0} |j| \langle x_j, y_j \rangle - 2\pi \sum_{j < 0} |j| \langle x_j, y_j \rangle.\end{aligned}$$

Recall that the inner product on the space $H^s(S^1)$ is defined in terms of Fourier series by

$$\langle x, y \rangle_s = \langle x_0, y_0 \rangle + 2\pi \sum_{k \in \mathbb{Z}} |k|^{2s} \langle x_k, y_k \rangle.$$

Therefore, the bilinear form

$$a(x, y) = \int_0^1 \frac{1}{2} \langle -J\dot{x}, y \rangle dt$$

can be defined as a continuous bilinear form on the Sobolev space $E = H^{1/2}(S^1)$. From now on, we write

$$\langle \ , \ \rangle = \langle \ , \ \rangle_{\frac{1}{2}} \text{ and } \| \ \| = \| \ \|_{\frac{1}{2}}.$$

There is an orthogonal splitting

$$E = E^- \oplus E^0 \oplus E^+$$

into spaces of $x \in E$ having only Fourier coefficients for $j < 0$, $j = 0$, $j > 0$. The corresponding orthogonal projections are denoted by $P^-$, $P^0$, $P^+$. Then we define for $x, y \in E$

$$\begin{aligned}a(x, y) &= \frac{1}{2} \langle x^+, y^+ \rangle - \frac{1}{2} \langle x^-, y^- \rangle \\ &= \frac{1}{2} \langle (P^+ - P^-)x, y \rangle.\end{aligned}$$

The function $a : E \to \mathbb{R}$ defined by

$$a(x) = a(x, x) = \frac{1}{2}\|x^+\|^2 - \frac{1}{2}\|x^-\|^2$$

is (Fréchet) differentiable with (Fréchet) derivative

$$da(x)(y) = \langle (P^+ - P^-)x, y\rangle$$

so the gradient of $a$ is

$$\nabla a(x) = (P^+ - P^-)x = x^+ - x^-.$$

We now study the function

$$b(x) = \int_0^1 H(x(t))dt,$$

recalling that $H$ vanishes near the origin. Since $|H(z)| \leq M|z|^2$ for $z \in \mathbb{R}^{2n}$, the map $b$ is defined for $x \in L^2$ and hence also for $x \in E \subset L^2$. If we consider $b$ as a function on $L^2$, we shall denote it by $\hat{b}$, so with the inclusion map $j : E \to L^2$, we have $b(x) = \hat{b}(j(x))$ for $x \in E$. To show that $\hat{b}$ is differentiable, we shall use the identity

$$H(z + \xi) = H(z) + \langle \nabla H(z), \xi\rangle + \int_0^1 \langle \nabla H(z + t\xi) - \nabla H(z), \xi\rangle dt.$$

Since $|H_{zz}(z)| \leq M$, the last term is $\leq M|\xi|^2$. Therefore, given $x, h \in L^2$, we have

$$\hat{b}(x + h) = \hat{b}(x) + \int_0^1 \langle \nabla H(x), h\rangle dt + R(x, h)$$

where $|R(x, h)| \leq M\|h\|^2_{L^2}$. This shows that $\hat{b}$ is differentiable with derivative given by

$$d\hat{b}(x)(h) = \int_0^1 \langle \nabla H(x), h\rangle dt = (\nabla H(x), h)_{L^2}.$$

Hence the $L^2$ gradient $\nabla\hat{b}(x) = \nabla H(x) \in L^2$. The derivative of $b : E \to \mathbb{R}$ is given by

$$db(x)(y) = d\hat{b}(j(x))(j(y))$$

and hence

$$\nabla b(x) = j^*\nabla\hat{b}(j(x)) = j^*\nabla H(x).$$

**Lemma 7.** *The gradient $\nabla b : E \to E$ is continuous and maps bounded sets to relatively compact sets. Moreover,*

$$\|\nabla b(x) - \nabla b(y)\| \leq M\|x - y\| \text{ and } |b(x)| \leq M\|x\|^2_{L^2} \text{ for } x, y \in E.$$

*Proof.* The map $x \mapsto \nabla H(x)$ is globally Lipschitz continuous on $L^2$, since $|\nabla H(z)| \leq M|z|$ for $z \in \mathbb{R}^{2n}$. Therefore, it maps bounded sets to bounded sets. The first claim then follows from the fact that $j^* : L^2 \to E$ is compact. Moreover,

$$\begin{aligned}\|\nabla b(x) - \nabla b(y)\|_{\frac{1}{2}} &= \|j^*(\nabla H(x) - \nabla H(y))\|_{\frac{1}{2}}\\ &\leq \|\nabla H(x) - \nabla H(y)\|_{L^2} \leq M\|x - y\|_{L^2}\\ &\leq M\|x - y\|_{\frac{1}{2}}.\end{aligned}$$

Finally, the last estimate follows from $|H(z)| \leq M|z|^2$. □

Summarizing the discussion so far, we have extended the action functional $\Phi$ from the space $\Omega$ of smooth loops to the the Hilbert space $E \supset \Omega$ by

$$\Phi(x) = a(x) - b(x) \text{ for } x \in E.$$

The function $\Phi : E \to \mathbb{R}$ is differentiable and its gradient is given by

$$\nabla\Phi(x) = x^+ - x^- - \nabla b(x).$$

As one should expect, there is a regularity statement for critical points of $\Phi$.

**Lemma 8.** *If $\nabla\Phi(x) = 0$, then $x \in C^\infty(S^1)$, and it solves the Hamiltonian equation.*

*Proof.* Represent $x$ and $\nabla H(x)$ by their Fourier series

$$x = \sum e^{k2\pi Jt} x_k$$
$$\nabla H(x) = \sum e^{k2\pi Jt} a_k.$$

By assumption, $d\Phi(x)(v) = 0$. This means that

$$\langle x^+ - x^-, v\rangle = \int_0^1 \langle \nabla H(x), v\rangle dt$$

for all $v \in E$. By choosing test functions $v(t)$, we find that

$$2\pi k x_k = a_k, \ k \in \mathbb{Z} \text{ and } a_0 = 0.$$

This implies that

$$\sum |k|^2 |x_k|^2 \le \sum |a_k|^2 < \infty,$$

so $x \in H^1$. By the Sobolev embedding theorem, this implies that $x \in C^0$. Consequently,

$$\xi(t) = \int_0^t J\nabla H(x(s)) ds$$

is in $C^1(\mathbb{R})$. By comparing Fourier coefficients, we see that $x(t) = x(0) + \xi(t)$. Thus $x \in C^1$ and solves the Hamiltonian equation $\dot{x} = J\nabla H(x)$. Once the right hand side is seen to be $C^1$, we infer that $x \in C^2$, and iterating this argument we see that $x \in C^\infty$. □

**Lemma 9.** *Every sequence $x_j \in E$ satisfying $\nabla\Phi(x_j) \to 0$ has a convergent subsequence. In particular, $\Phi$ satisfies the P.S. condition.*

*Proof.* Assume that $\nabla\Phi(x_j) = x_j^+ - x_j^- - \nabla b(x_j) \to 0$. We claim that it suffices to show that $x_j$ is bounded in $E$. If so, since $\nabla b$ is compact, we infer that $x_j^+ - x_j^-$ has a convergent subsequence, and by orthogonality $x_j^+$ and $x_j^-$ must each have a convergent subsequence. But $x_j^0 \in \mathbb{R}^{2n}$ is also bounded, and thus has a convergent subsequence.

To show that $x_j$ is bounded we argue by contradiction and assume that $\|x_j\| \to \infty$. Consider $y_k = x_k/\|x_k\|$. Then

$$y_k^+ - y_k^- - j^*\left(\frac{1}{\|x_k\|}\nabla H(x_k)\right) \to 0.$$

Since $|\nabla H(z)| \le M|z|$ the sequence $\frac{\nabla H(x_k)}{\|x_k\|}$ is bounded in $L^2$, and since $j^* : L^2 \to E$ is compact, $y_k^+ - y_k^-$ is relatively compact, hence $y_k$ is relatively compact. After passing to a subsequence we may assume that $y_k \to y$ in $E$ and hence $y_k \to y$ in $L^2$. Recall that $|\nabla H(z) - \nabla Q(z)| \le M$ for some quadratic $Q$.

$$\left\|\frac{\nabla H(x_k)}{\|x_k\|} - \nabla Q(y)\right\|_{L^2} \le \frac{1}{\|x_k\|}\left\|\nabla H(x_k) - \nabla Q(x_k)\right\|_{L^2} + \left\|\nabla Q(y_k - y)\right\|_{L^2},$$

and hence

$$\frac{\nabla H(x_k)}{\|x_k\|} \to \nabla Q(y) \text{ in } L^2.$$

Consequently,

$$\frac{\nabla b(x_k)}{\|x_k\|} \to j^*(\nabla Q(y)) \text{ in } E,$$

and hence $y \in E$ satisfies

$$y^+ - y^- - j^*(\nabla Q(y)) = 0.$$

By the regularity argument from the previous lemma, we deduce that $y \in C^\infty$ is a 1-periodic orbit of $X_Q$. But all nontrivial orbits of $X_Q$ have period $T \neq 1$, so $y(t) = 0$, which contradicts $\|y\| = 1$. □

The gradient equation of $\Phi$ is globally Lipschitz continuous, and hence defines a unique global flow $\varphi^t$ by ODE theory.

**Lemma 10.** *The flow of $\dot{x} = -\nabla\Phi(x)$ can be written as*

$$\varphi^t(x) = e^t x^- + x^0 + e^{-t}x^+ + K(t,x)$$

*where $K : \mathbb{R} \times E \to E$ is continuous and maps bounded sets to relatively compact sets.*

*Proof.* Define

$$K(t,x) = -\int_0^t (e^{t-s}P^- + P^0 + e^{s-t}P^+)\nabla b(\varphi^s(x))ds.$$

It has the desired properties since $j^* : L^2 \to E$ is compact. To verify the equality, let us denote the right hand side by $y(t)$ and differentiate

$$\dot{y}(t) = (P^- - P^+)y(t) - \nabla b(\varphi^t(x)).$$

Since $y(0) = x$, the function $\xi(t) = y(t) - \varphi^t(x)$ satisfies the equation

$$\dot{\xi}(t) = (P^- - P^+)\xi(t)$$

with initial value $\xi(0) = 0$. By the uniqueness theorem in ODE theory, $\xi(t) = 0$. □

We are now prepared to prove the existence of the desired 1-periodic orbit.

We first single out two distinguished subsets $\Sigma$ and $\Gamma$ of $E$. Define

$$\Sigma = \Sigma_\tau = \{x \mid x = x^- + x^0 + se^+,\ \|x^- + x^0\| \le \tau,\ 0 \le s \le \tau\}$$

where $\tau > 0$, and $e^+(t) = e^{2\pi Jt}e_1$, $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{2n}$. Clearly $\|e^+\|^2 = 2\pi$ and $\|e^+\|_{L^2} = 1$. We denote by $\partial\Sigma$ the boundary of $\Sigma$ in $E^- + E^0 + \mathbb{R}e^+$.

**Lemma 11.** *There exists $\tau^+ > 0$ such that for all $\tau \ge \tau^*$*

$$\Phi|_{\partial\Sigma} \le 0.$$

*Proof.* By definition $\Phi|_{E^-\oplus E^0} \le 0$. Thus it remains to consider the cases $\|x^- + x^0\| = \tau$ and $s = \tau$. By the construction of $H$ there exists a constant $\gamma > 0$ such that

$$H(z) \ge (\pi - \varepsilon)q(z) - \gamma.$$

Then

$$\Phi(x) \le a(x) - (\pi - \varepsilon)\int_0^1 q(x) + \gamma$$

for all $x \in E$. We can estimate for $x = x^- + x^0 + se^+$

$$\begin{aligned}\Phi(x) &\le \frac{1}{2}s^2\left\|e^+\right\|^2 - \frac{1}{2}\left\|x^-\right\|^2 - (\pi+\varepsilon)q\left(x^0\right) - (\pi+\varepsilon)\int_0 q\left(se^+\right) + \gamma \\ &= -\frac{1}{2}\left\|x^-\right\|^2 - \varepsilon s^2\left\|e^+\right\|_{L^2}^2 - (\pi+\varepsilon)q\left(x^0\right) + \gamma.\end{aligned}$$

Therefore, there exists $c > 0$ such that

$$\Phi(x) \le \gamma - c\|x^- + x^0\|^2 - c\|se^+\|^2.$$

The right hand side will be negative if $\|x^- + x^0\|$ or $s$ becomes sufficiently large. □

Define

$$\Gamma = \Gamma_\alpha = \{x \in E^+ \mid \|x\| = \alpha\}.$$

**Lemma 12.** *There exist $\alpha, \beta > 0$ such that*

$$\Phi|_\Gamma \ge \beta > 0.$$

*Proof.* By the Sobolev embedding theorem, for $p \ge 1$, there exist constants $M = M_p$ such that

$$\|u\|_{L^p} \le M\|u\|_{1/2},\ u \in H^{1/2}(S^1).$$

Since $|H(z)| \leq c|z|^3$, there exists a constant $K > 0$ such that

$$\int_0^1 |H(x(t))|dt \leq c\|x\|_{L^3}^3 \leq K\|x\|_{1/2}^3$$

for all $x \in H^{1/2}$. If $x \in E^+$, then

$$\Phi(x) \geq \frac{1}{2}\|x\|^2 - K\|x\|^3$$

and the claim follows. □

See Figure 2.

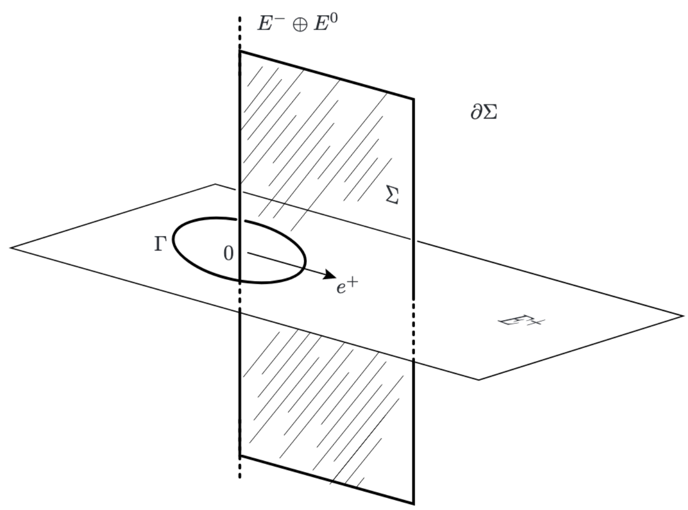


FIGURE 2. Reproduced from [4] for educational purposes only.

Now since $\Phi(\varphi^t(x))$ decreases in $t$, we deduce from Lemma 11 that

$$\Phi|_{\varphi^t(\partial\Sigma)} \leq 0$$

for all $t \geq 0$. In view of Lemma 12, we have $\varphi^t(\partial\Sigma) \cap \Gamma = \emptyset$ for all $t \geq 0$. Then intuitively, it is clear that $\varphi^t(\Sigma) \cap \Gamma \neq \emptyset$ for all $t \geq 0$.

**Lemma 13.** *$\varphi^t(\Sigma) \cap \Gamma \neq \emptyset$ for all $t \geq 0$.*

*Proof.* Abbreviate the flow by $\varphi^t(x) = x \cdot t$. We wish to find $x \in \Sigma$ such that

$$\left(P^- + P^0\right)(x \cdot t) = 0$$
$$\|x \cdot t\| = \alpha.$$

Using Lemma 10, the equations become

$$0 = e^t x^- + x^0 + \left(P^- + P^0\right) K(t, x)$$
$$0 = \alpha - \|x \cdot t\|.$$

Multiplying the $E^-$ part by $e^{-t}$, we obtain the equivalent equations

$$0 = x^- + x^0 + \left(e^{-t} P^- + P^0\right) K(t, x)$$
$$0 = \alpha - \|x \cdot t\|.$$

For $x = x^- + x^0 + se^+$, this can be further rewritten as

$$0 = x + B(t, x)$$

where the operator $B$ is defined by

$$B(t,x) = \left(e^{-t}P^- + P^0\right)K(t,x) + P^+\left\{(\|x\cdot t\| - \alpha)e^+ - x\right\}.$$

Abbreviating $F = E^- \oplus E^0 \oplus \mathbb{R}e^+$ the map $B : \mathbb{R}\times F \to F$ is continuous and maps bounded sets to relatively compact sets (Lemma 10). Hence Leray-Schauder index theory applies. To find $x \in \Sigma$ such that $x + B(t,x) = 0$, it suffices to show that

$$\deg(\Sigma, \mathrm{id} + B(t,\cdot), 0) \neq 0.$$

Since $\varphi^t(\partial\Sigma) \cap \Gamma = \emptyset$ for all $t \geq 0$, there is no solution on $\partial\Sigma$. Hence by the homotopy invariance of the degree,

$$\deg(\Sigma, \mathrm{id} + B(t,\cdot), 0) = \deg(\Sigma, \mathrm{id} + B(0,\cdot), 0).$$

Since $K(0,x) = 0$ we have $B(0,x) = P^+\{(\|x\| - \alpha)e^+ - x\}$. Consider the homotopy

$$L_\mu(x) = P^+\left\{(\mu\|x\| - \alpha)e^+ - \mu x\right\}$$

for $\mu \in [0,1]$. If $x \in \Sigma$ satisfies $x + L_\mu(x) = 0$, then $x = se^+$, and $s(1 - \mu + \mu\|e^+\|) = \alpha$. Consequently, $0 < s < \alpha$, and thus $x \notin \partial\Sigma$ if $\tau > \alpha$. Therefore, we can again apply the homotopy invariance of degree

$$\begin{aligned}\deg(\Sigma, \mathrm{id} + B(0,\cdot), 0) &= \deg(\Sigma, \mathrm{id} - \alpha e^+, 0)\\ &= \deg(\Sigma, \mathrm{id}, \alpha e^+).\end{aligned}$$

If $\tau > \alpha$, then $\alpha e^+ \in \Sigma$, and hence the degree is 1. □

We now take the family $\mathcal{F} = \{\varphi^t(\Sigma)\}_{t\geq 0}$, and consider

$$c(\Phi, \mathcal{F}) = \inf_{t\geq 0} \sup_{x\in\varphi^t(\Sigma)} \Phi(x).$$

Since $\varphi^t(\Sigma) \cap \Gamma \neq \emptyset$ (Lemma 13) and $\Phi|_\Gamma \geq \beta$ (Lemma 12), we have that

$$\beta \leq \inf_{x\in\Gamma} \Phi(x) \leq \sup_{x\in\varphi^t(\Sigma)} \Phi(x) < \infty.$$

The last inequality uses Lemma 7, which implies that $\Phi$ maps bounded sets to bounded sets. Therefore

$$c(\Phi, \mathcal{F}) \in [\beta, \infty).$$

By Lemma 9, $\Phi$ satisfies the P.S. condition, and by Lemma 7, its gradient defines a unique global flow. Consequently, we can apply Lemma 6, concluding that $c(\Phi, \mathcal{F})$ is a critical value. This means that there exists $x^* \in E$ satisfying $\nabla\Phi(x^*) = 0$ and $\Phi(x^*) = c(\Phi, \mathcal{F}) \geq \beta > 0$. By Lemma 8, $x^*$ is a smooth 1-periodic orbit of $X_H$.

The proof is thus complete.

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